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Oxford Cambridge and RSA

Wednesday 07 October 2020 – Afternoon

A Level Mathematics A

H240/01 Pure Mathematics

Time allowed: 2 hours

You must have:
- the Printed Answer Booklet
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **100**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

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Formulae
A Level Mathematics A (H240)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$$

where $ {}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1, n \in \mathbb{R})$$

Differentiation

$f(x)$	$f'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient rule $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Integration

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)(f(x))^n\,dx = \frac{1}{n+1}(f(x))^{n+1} + c$$

Integration by parts $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A)P(B \mid A) = P(B)P(A \mid B)$ **or** $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

Standard deviation

$\sqrt{\dfrac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\dfrac{\Sigma x^2}{n} - \bar{x}^2}$ **or** $\sqrt{\dfrac{\Sigma f(x-\bar{x})^2}{\Sigma f}} = \sqrt{\dfrac{\Sigma fx^2}{\Sigma f} - \bar{x}^2}$

The binomial distribution

If $X \sim B(n, p)$ then $P(X = x) = \dbinom{n}{x} p^x (1-p)^{n-x}$, mean of X is np, variance of X is $np(1-p)$

Hypothesis test for the mean of a normal distribution

If $X \sim N(\mu, \sigma^2)$ then $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $P(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Kinematics

Motion in a straight line	Motion in two dimensions
$v = u + at$	$\mathbf{v} = \mathbf{u} + \mathbf{a}t$
$s = ut + \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$
$s = \frac{1}{2}(u+v)t$	$\mathbf{s} = \frac{1}{2}(\mathbf{u}+\mathbf{v})t$
$v^2 = u^2 + 2as$	
$s = vt - \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Answer **all** the questions.

1 **(a)** For a small angle θ, where θ is in radians, show that $2\cos\theta + (1-\tan\theta)^2 \approx 3 - 2\theta$. **[3]**

(b) Hence determine an approximate solution to $2\cos\theta + (1-\tan\theta)^2 = 28\sin\theta$. **[2]**

a) $\cos\theta \approx 1 - \frac{1}{2}\theta^2$

$\tan\theta \approx \theta$

$2(1 - \frac{1}{2}\theta^2) + (1-\theta)^2$

$2 - \cancel{\theta^2} + 1 - 2\theta + \cancel{\theta^2}$

$\Rightarrow 3 - 2\theta$ as required

b) $3 - 2\theta = 28\theta$

$30\theta = 3$

$\theta = \frac{1}{10} = 0.1$

2 Simplify fully.

Laws of indices

$a^m \times a^n = a^{m+n}$ (1)

$a^m \div a^n = a^{m-n}$ (2)

$(a^m)^n = a^{m \times n}$ (3)

(a) $\sqrt{12a} \times \sqrt{3a^5}$ **[2]**

(b) $(64b^3)^{\frac{1}{3}} \times (4b^4)^{-\frac{1}{2}}$ **[2]**

(c) $7 \times 9^{3c} - 4 \times 27^{2c}$ **[4]**

a) $12 = 4 \times 3$

$\sqrt{4} \times \sqrt{3a} = 2(3a)^{1/2} \times (3a^5)^{1/2}$

(3) $2 \times 3^{1/2} \times a^{1/2} \times 3^{1/2} \times a^{5/2}$

$1/2 + 5/2 = 6/2 = 3$ $\quad 3^{1/2} \times 3^{1/2}$ $\quad 1/2 + 1/2 = 1 \quad \therefore 3^1 = 3$ (1)

$2 \times 3 \times a^3 = \boxed{6a^3}$

b) $64^{1/3} \times (b^3)^{1/3} \times 4^{-1/2} \times (b^4)^{-1/2}$

(3) $4 \times b^1 \times \frac{1}{2} \times b^{-2}$

$\Rightarrow 4 \times \frac{1}{2} \times b^1 \times b^{-2}$ $\quad 1 + -2 = -1$ (1)

$= 2 \times b^{-1} = \boxed{\frac{2}{b}}$

c) Note: $3^2 = 9$ and $3^3 = 27$

$\Rightarrow 7 \times (3^2)^{3c} - 4(3^3)^{2c}$ (3)

$\Rightarrow 7 \times 3^{6c} - 4 \times 3^{6c}$

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$\Rightarrow 7-4=3$

3×3^{6c}

Remember $\Rightarrow 3 = 3^1$

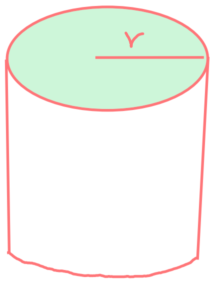
$\therefore 3^1 \times 3^{6c}$ ①

$\Rightarrow \boxed{3^{6c+1}}$

3 A cylindrical metal tin of radius r cm is closed at both ends. It has a volume of $16000\pi\text{ cm}^3$.

(a) Show that its total surface area, $A\text{ cm}^2$, is given by $A = 2\pi r^2 + 32000\pi r^{-1}$. **[4]**

(b) Use calculus to determine the minimum total surface area of the tin. You should justify that it is a minimum. **[6]**



Volume $= 16000\pi$

Remember Volume

$=$ CSA $\times$ height

CSA $= \pi r^2$ (circle)

$$16000\pi = \pi r^2 \times h.$$

$$h = \frac{16000\cancel{\pi}}{\cancel{\pi} r^2} = \frac{16000}{r^2}$$

$\therefore$ Surface area $=$ Circle $+$ circle $+$ The wrap around (rectangle)

$\Rightarrow \pi r^2 + \pi r^2 +$ circumference (height)

$\Rightarrow 2\pi r^2 + 2\pi r(h)$ from above

$$A = 2\pi r^2 + 2\pi \cancel{r}\left(\frac{16000}{\cancel{r^2}\, r}\right)$$

$$A = 2\pi r^2 + \frac{32000\pi}{r} = 2\pi r^2 + 32000\pi r^{-1}$$ as req.

b) $\frac{dA}{dr} = 4\pi r + 32000(-1)\pi r^{-2}$

$= 4\pi r - 32000\pi r^{-2}$

$\frac{dA}{dr} = 0$ @ minimum value of r

$0 = 4\pi r - \frac{32000\pi}{r^2}$

$\frac{r^2}{\pi} \times \quad \frac{32000\pi}{r^2} = 4\pi r \quad \times \frac{r^2}{\pi}$

$4 \div \quad 32000 = 4r^3 \quad \div 4$

$r^3 = 8000$

$r = \sqrt[3]{8000} = 20$

Justification

$\frac{d^2A}{dr^2} = 4\pi - 32000(-2)\pi r^{-3}$

$= 4\pi + 64000\pi r^{-3}$

when $r = 20$

$4\pi + 64000\pi(20)^{-3} > 0$ $\therefore$ Value of r is minimum

$\therefore$ Minimum SA $= 2\pi(20)^2 + 32000\pi(20)^{-1}$
from (a)

$= 7540 cm^2$

4 Prove by contradiction that there is no greatest multiple of 5. **[3]**

Step 1

Assume that there is a greatest multiple of 5 $= 5 \times K = 5K$ where K is an integer

Step 2

If we add 5 to this greatest multiple of 5 $\Rightarrow 5K + 5$

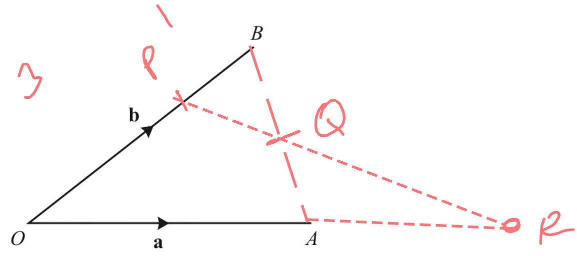
Step 3

$5(K+1)$; since K was an integer this is also a multiple of 5.

Conclusion

since $K+1 > K$, $5(K+1) > 5$, this contradicts the assumption $\therefore$ there is no greatest multiple of 5

5



The diagram shows points A and B, which have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to an origin O. P is the point on OB such that $OP : PB = 3:1$ and Q is the midpoint of AB.

(a) Find $\overrightarrow{PQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. **[2]**

The line OA is extended to a point R, so that PQR is a straight line.

(b) Explain why $\overrightarrow{PR} = k\,(2\mathbf{a} - \mathbf{b})$, where k is a constant. **[2]**

(c) Hence determine the ratio $OA : AR$. **[4]**

a) $\overrightarrow{OP} = \frac{3}{4}(\overrightarrow{OB}) = \frac{3}{4}(b) = \frac{3}{4}\underset{\sim}{b}$

$\overrightarrow{AQ} = \frac{1}{2}(\overrightarrow{AB})$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\underset{\sim}{a} + \underset{\sim}{b}$

$\therefore \overrightarrow{AQ} = \frac{1}{2}(-\underset{\sim}{a} + \underset{\sim}{b}) = -\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$

$\therefore \overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OA} + \overrightarrow{AQ}$

$= -\frac{3}{4}\underset{\sim}{b} + \underset{\sim}{a} + \left(-\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}\right)$

$= -\frac{3}{4}\underset{\sim}{b} + \underset{\sim}{a} - \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} = \frac{1}{2}\underset{\sim}{a} - \frac{1}{4}\underset{\sim}{b}$

$= \frac{1}{4}(2a - b)$

b) $\overrightarrow{PR}$ is in the same direction as $\overrightarrow{PQ}$ (ie they are parallel) $\therefore$ $\overrightarrow{PR} = k\overrightarrow{PQ}$ ie a multiple of $\overrightarrow{PQ}$

i.e. $\overrightarrow{PR} = \frac{1}{4}\lambda(2a-b) = k(2a-b)$

c) $\overrightarrow{AR} = \overrightarrow{AO} + \overrightarrow{OP} + \overrightarrow{PR}$

$= -a + \frac{3}{4}b + k(2a-b)$

NB: We didn't include 1/4 as that is also a constant

$= -a + \frac{3}{4}b + 2ak - kb$

Since $\overrightarrow{AR}$ is parallel to $\overrightarrow{OA}$, the component of the vector that is b is equal to 0, as there is no b component in $\overrightarrow{OA}$

$\frac{3}{4}b - kb = 0$

$k = 3/4$

$\therefore \overrightarrow{AR} = -a + 2(3/4)a = \frac{1}{2}a$

$\therefore \overrightarrow{OA} : \overrightarrow{AR}$

$1 : 1/2 = 2:1$

6 A mobile phone company records their annual sales on 31st December every year.

Paul thinks that the annual sales, S million, can be modelled by the equation $S = ab^t$, where a and b are both positive constants and t is the number of years since 31st December 2015.

Paul tests his theory by using the annual sales figures from 31st December 2015 to 31st December 2019. He plots these results on a graph, with t on the horizontal axis and $\log_{10} S$ on the vertical axis.

(a) Explain why, if Paul's model is correct, the results should lie on a straight line of best fit on his graph. **[3]**

The results lie on a straight line of best fit which has a gradient of 0.146 and an intercept on the vertical axis of 0.583.

(b) Use these values to obtain estimates for a and b, correct to 2 significant figures. **[2]**

(c) Use this model to predict the year in which, on the 31st December, the annual sales would first be recorded as greater than 200 million. **[3]**

(d) Give a reason why this prediction may not be reliable. **[1]**

a) $S = ab^t$ applying log to both sides

$\Rightarrow \log S = \log(ab^t)$

$\Rightarrow \log S = \log a + \log(b^t)$

$\Rightarrow \log S = \log a + t\log b$

$\Rightarrow \log S = t\log b + \log a$

comparable to $y = mx + c$

$\therefore$ should lie on a straight line

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b) $\log a = c$

$\log a = 0.583$

$10^{0.583} = a = 3.8$

$\log b = 0.146$

$10^{0.146} = b = 1.4$

c) $S = ab^t$

$S = 3.8 \times 1.4^t$

$3.8 \times 1.4^t = 200$

$1.4^t = \dfrac{200}{3.8}$

$t \log 1.4 = \log\left(\dfrac{200}{3.8}\right)$

$t = \dfrac{\log(200/3.8)}{\log(1.4)} = 11.8$

$\therefore 2015 + 11.8 = 2026.8 \quad \therefore$ Year $= 2027$

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d) → It is unlikely that the sales will continue at the same rate

→ There is only a finite market

7 Two students, Anna and Ben, are starting a revision programme. They will both revise for 30 minutes on Day 1. Anna will increase her revision time by 15 minutes for every subsequent day. Ben will increase his revision time by 10% for every subsequent day.

(a) Verify that on Day 10 Anna does 94 minutes more revision than Ben, correct to the nearest minute. **[3]**

Let Day X be the first day on which Ben does more revision than Anna.

(b) Show that X satisfies the inequality $X > \log_{1.1}(0.5X + 0.5) + 1$. **[3]**

(c) Use the iterative formula $x_{n+1} = \log_{1.1}(0.5x_n + 0.5) + 1$ with $x_1 = 10$ to find the value of X.

You should show the result of each iteration. **[3]**

(d) **(i)** Give a reason why Anna's revision programme may not be realistic. **[1]**

(ii) Give a **different** reason why Ben's revision programme may not be realistic. **[1]**

a) Anna

Arithmetic series

$a = 30$ $\quad a + d(n-1) \rightarrow n^{th}$ term

$d = 15$

$n = 10$ $\quad = 30 + 15(10-1)$

$= 165$

Ben

Geometric series

$a = 30$

$d = 1.1 \rightarrow 100\% + 10\% = 110\%$

n^{th} term $\rightarrow ar^{n-1}$

$30 \times 1.1^{10-1} = 70.7$

$165 - 70.7 = 94.3$

$\therefore$ 94 mins to the nearest minute as required

b) Day x is the 1st Day Ben does more revision than Anna

Anna: $U_x = 30 + 15(x-1)$

Ben: $U_x = 30 \times 1.1^{x-1}$

$30 \times 1.1^{x-1} > 30 + 15(x-1)$

$30 \times 1.1^{x-1} > 30 + 15x - 15$

$30 \times 1.1^{x-1} > 15 + 15x$ $\quad (2+x)$

$\frac{\cancel{30} \times 1.1^{x-1}}{\cancel{30}} > \frac{1\cancel{5}(1+x)}{\cancel{30}\ 2}$

$1.1^{x-1} > \frac{1}{2}(1+x)$

$(x-1)\cancel{\log_{1.1} 1.1} > \log_{1.1}\left(\frac{1}{2} + \frac{1}{2}x\right)$

1

$$x-1 > \log_{1.1}\left(\tfrac{1}{2}+\tfrac{1}{2}x\right)$$

$$x > \log_{1.1}\left(\tfrac{1}{2}+\tfrac{1}{2}x\right)+1$$

as required.

c) $x_{n+1} = \log_{1.1}\left(\tfrac{1}{2}+\tfrac{1}{2}x_n\right)+1$

$x_1 = 10$

$x_2 = \log_{1.1}\left(\tfrac{1}{2}+\tfrac{1}{2}(10)\right)+1 = 18.9$

$x_3 = \log_{1.1}\left(\tfrac{1}{2}+\tfrac{1}{2}(18.9)\right)+1 = 25.1$

$\vdots$

29.4

29.6

29.6

$\therefore x = 30$

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di) → Eventually there will not be enough hours for revision

ii) → Increasing by 10% each time will eventually involve decimals ∴ time will not be accurate

8 **(a)** Differentiate $(2+3x^2)e^{2x}$ with respect to x. **[3]**

(b) Hence show that $(2+3x^2)e^{2x}$ is increasing for all values of x. **[4]**

a) $\underbrace{(2+3x^2)}_{u}\underbrace{e^{2x}}_{v}$

Using Chain Rule

$uv' + vu'$

$v = e^{kx}$ $v' = ke^{kx}$

$u = 2+3x^2$

$u' = 6x$

$v = e^{2x}$

$v' = 2e^{2x}$

$\Rightarrow (2+3x^2)(2e^{2x}) + e^{2x}(6x)$

$e^{2x}\left[2(2+3x^2) + 6x\right]$

$\Rightarrow e^{2x}\left[4+6x^2+6x\right]$

$\Rightarrow e^{2x}(6x^2+6x+4)$

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b) $e^{2x} > 0$ for all values of x

completing the square for;

$6x^2 + 6x + 4$

$6\left[x^2 + \frac{6x}{6} + \frac{4}{6}\right]$

$\Rightarrow 6\left[\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + \frac{2}{3}\right]$

$\Rightarrow 6\left[(x + 1/2)^2 + 5/12\right]$

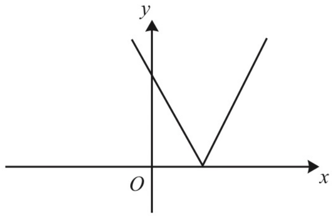
$\Rightarrow 6(x + 1/2)^2 + 5/2$

$\Rightarrow e^{2x}\left(\underbrace{6(x + 1/2)^2 + 5/2}\right)$

min value of $f(x)$ is $5/2$ which is greater than 0 for all values of x

$\therefore$ The gradient of $e^{2x}\left(6(x+1/2)^2 + 5/2\right)$ is greater than 0 for all values of x so it is increasing for all values of x

9



The diagram shows the graph of $y = |2x - 3|$.

(a) State the coordinates of the points of intersection with the axes. **[2]**

(b) Given that the graphs of $y = |2x - 3|$ and $y = ax + 2$ have two distinct points of intersection, determine

(i) the set of possible values of a, **[4]**

(ii) the x-coordinates of the points of intersection of these graphs, giving your answers in terms of a. **[3]**

a) 1 X int $y = 0$

$0 = |2x - 3|$

$\therefore 2x - 3 = 0$

$x = 3/2$

$(3/2, 0)$

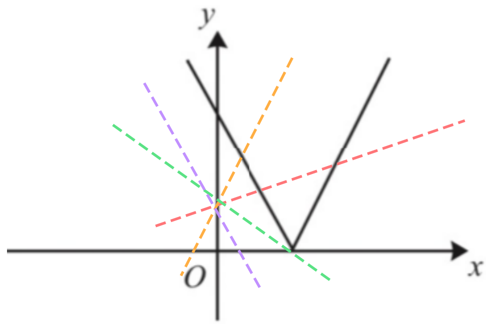
2 Y int $x = 0$

$y = |0 - 3|$

$y = |-3| = 3$

$\therefore (0, 3)$

b)



All the possible lines in the format $y = ax + 2$ have been drawn above. We will now go through all the possible cases

Line 1

→ This line has 2 points of intersection ∴ is a valid option.

→ The gradient of this line is less than the one of $y = 2x + 3$ ∴ $a < 2$ ⊛

Line 2

→ Not a valid option as only 1 POI → POI = Point of Intersection

→ occurs when $a = 2$

Line 3

→ This is a possible option as 2 POI are present above the x intercept of the graph. Let's find a at the x-int which was $3/2$.

$y = ax + 2$ $\quad 0 = 1.5a + 2$ $\quad a = -4/3$

$\therefore$ $a > -4/3$ there are 2 POI

Line 4

→ Not a possible option as only 1 POI, and occurs when $a = -2$

$\therefore$ From all of the above we can make a conclusion that;

$$-\frac{4}{3} < a < 2$$

ii) POI

$+(2x-3) = ax + 2$

$2x - ax = 5$

$x = \frac{5}{(2-a)}$

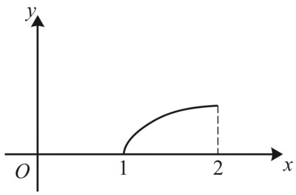
$-(2x-3) = ax + 2$

$-2x + 3 = ax + 2$

$2x + ax = 3 - 2$

$x = \frac{1}{(2+a)}$

10



The diagram shows the curve $y = \sin\left(\frac{1}{2}\sqrt{x-1}\right)$, for $1 \leqslant x \leqslant 2$.

(a) Use rectangles of width 0.25 to find upper and lower bounds for $\int_1^2 \sin\left(\frac{1}{2}\sqrt{x-1}\right)dx$. Give your answers correct to 3 significant figures. **[4]**

(b) (i) Use the substitution $t = \sqrt{x-1}$ to show that $\int \sin\left(\frac{1}{2}\sqrt{x-1}\right)dx = \int 2t\sin\left(\frac{1}{2}t\right)dt$. **[3]**

(ii) Hence show that $\int_1^2 \sin\left(\frac{1}{2}\sqrt{x-1}\right)dx = 8\sin\frac{1}{2} - 4\cos\frac{1}{2}$. **[4]**

a) Lower Bound.

$0.25(\sin 0 + \sin(\frac{1}{2}\sqrt{0.25}) + \sin(\frac{1}{2}\sqrt{0.5}) +$

$\sin(\frac{1}{2}\sqrt{0.75})$

$= 0.253$

Upper Bound

$0.25(\sin(\frac{1}{2}\sqrt{0.25}) + \sin(\frac{1}{2}\sqrt{0.5}) + \sin(\frac{1}{2}\sqrt{0.75})$

$+ \sin \frac{1}{2})$

$= 0.373$

bi) $t = \sqrt{x-1}$

$t^2 = x - 1$

$2t \cdot dt = 1\ dx$

$2t\ dt = dx$

$$\int \sin\left(\tfrac{1}{2}\sqrt{x-1}\right) dx \Rightarrow \int \sin(\tfrac{1}{2}t) \cdot 2t \cdot dt$$

$$\Rightarrow \int 2t \sin(\tfrac{1}{2}t)\ dt$$

as req.

ii) $\int_{t_1}^{t_2} 2t \sin(\tfrac{1}{2}t)\ dt.$

Integration By Parts

$u = 2t \qquad v' = \sin(\tfrac{1}{2}t)$

$u' = 2 \qquad v = \dfrac{-\cos(\tfrac{1}{2}t)}{\tfrac{1}{2}} = -2\cos(\tfrac{1}{2}t)$

$$uv - \int v \frac{du}{dx}\, dx$$

$$2t\left(-2\cos\left(\tfrac{1}{2}t\right)\right) - \int \left[2 \cdot -2\cos\left(\tfrac{1}{2}t\right)\right] dt$$

$$-4t\cos\left(\tfrac{1}{2}t\right) + 4\int \cos\left(\tfrac{1}{2}t\right) dt$$

$$\downarrow$$

$$4\left[\frac{\sin\left(\tfrac{1}{2}t\right)}{\tfrac{1}{2}}\right] \Rightarrow 8\sin\left(\tfrac{1}{2}t\right)$$

$$\Rightarrow \left[-4t\cos\left(\tfrac{1}{2}t\right) + 8\sin\left(\tfrac{1}{2}t\right)\right]_{t_1}^{t_2}$$

$t = \sqrt{x-1}$ (from above)

$t = \sqrt{2-1} = 1$

$t = \sqrt{1-1} = 0$

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$$\Rightarrow \left[-4t\cos\left(\tfrac{1}{2}t\right) + 8\sin\left(\tfrac{1}{2}t\right)\right]_0^1$$

$$\Rightarrow -4\cos\left(\tfrac{1}{2}\right) + 8\sin\left(\tfrac{1}{2}\right) - \left[0 + 0\right]$$

$$\Rightarrow 8\sin\tfrac{1}{2} - 4\cos\tfrac{1}{2} \quad \text{as required.}$$

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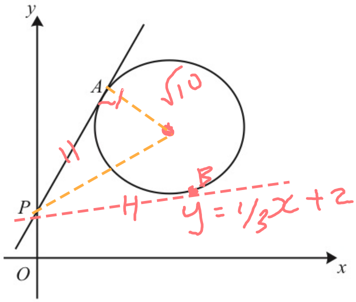
11 In this question you must show detailed reasoning.



The diagram shows a circle with equation $x^2 + y^2 - 10x - 14y + 64 = 0$. A tangent is drawn from the point $P(0, 2)$ to meet the circle at the point A. The equation of this tangent is of the form $y = mx + 2$, where m is a constant **greater than** 1.

(a) **(i)** Show that the x-coordinate of A satisfies the equation $(m^2 + 1)x^2 - 10(m + 1)x + 40 = 0$. **[2]**

(ii) Hence determine the equation of the tangent to the circle at A which passes through P. **[4]**

A second tangent is drawn from P to meet the circle at a second point B. The equation of this tangent is of the form $y = nx + 2$, where n is a constant **less than** 1.

(b) Determine the exact value of tan APB. **[4]**

ai) Since they have a common point of intersection;

$x^2 + (mx+2)^2 - 10x - 14(mx+2) + 64 = 0$

$x^2 + m^2x^2 + 4mx + 4 - 10x - 14mx - 28 + 64 = 0$

simplifying

$x^2 + m^2x^2 - 10mx + 40 - 10x = 0$

$x^2(1+m^2) - 10x(m+1) + 40 = 0$ as required

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ii) Using $b^2 - 4ac = 0$

$(-10)^2(m+1)^2 - 4(m^2+1)(40) = 0$

$100(m+1)^2 - 160m^2 - 160 = 0$

$\Rightarrow 100(m^2+2m+1) - 160m^2 - 160 = 0$

$100m^2 + 200m + 100 - 160m^2 - 160 = 0$

$-60m^2 + 200m - 60 = 0$

$$\frac{-200 \sqrt{(-200)^2 - 4(-60 \times -60)}}{2 \times -60}$$

$m = 3$ or $m = 1/3$

since $m > 1$ $\quad m = 3$

$y = 3x + 2$

b) ① Radius of the circle

$x^2 + y^2 - 10x - 14y + 64 = 0$

$(x-5)^2 - (-5)^2 + (y-7)^2 - (-7)^2 + 64 = 0$

$(x-5)^2 + (y-7)^2 - 10 = 0$

$(x-5)^2 + (y-7)^2 = 10$

$(x-a)^2 + (y-b)^2 = r^2$

where centre of circle $= (a, b)$

radius $= \sqrt{r^2} = r$.

$\therefore$ radius $= \sqrt{10}$

② Co-ordinates of A.

Using (ai) and $m = 3$

$x^2(1+3^2) - 10x(3+1) + 40 = 0$

$10x^2 - 40x + 40 = 0$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x = 2$.

$y = 3x + 2$

$y = 3(2) + 2$

$y = 8$ $\qquad \therefore A = (2, 8)$

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Length AP.

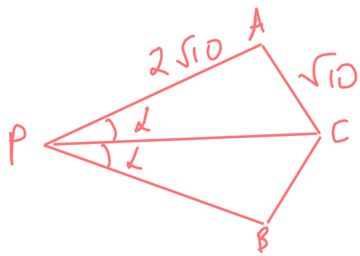
$\overset{x_1\ y_1}{(2,8)}$ $\overset{x_2\ y_2}{(0,2)}$

$$\text{Length} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$= \sqrt{(2-8)^2 + (0-2)^2}$$

$$= \sqrt{(-6)^2 + (-2)^2}$$

$$= 2\sqrt{10}$$



$$\tan APC = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{10}}{2\sqrt{10}} = \frac{1}{2}$$

$$\therefore \tan APB = \frac{1}{1-1/4} = 4/3$$

12 Find the general solution of the differential equation

$$(2x^3 - 3x^2 - 11x + 6)\frac{dy}{dx} = y(20x - 35).$$

Give your answer in the form $y = f(x)$. **[9]**

$$(2x^3 - 3x^2 - 11x + 6)\frac{dy}{dx} = y(20x-35)$$

$$\times \frac{dx}{y(2x^3-3x^2-11x+6)}$$ on both sides

$$\int \frac{1}{y}\,dy = \int \frac{20x-35}{\underbrace{(2x^3-3x^2-11x+6)}}\,dx$$

Need to factorise this

$f(x) = 2x^3 - 3x^2 - 11x + 6$

$f(3) = 0 \quad \therefore (x-3) \Rightarrow$ is a factor.

Using long division find the other 2 factors of $f(x)$

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$$
\begin{array}{r|l}
 & 2x^2 + 3x - 2 \\
\hline
(x-3) & 2x^3 - 3x^2 - 11x + 6 \\
 & -\underline{(2x^3 - 6x^2)} \\
 & 3x^2 - 11x \\
 & -\underline{(3x^2 - 9x)} \\
 & -2x + 6 \\
 & -\underline{(-2x + 6)} \\
 & 0
\end{array}
$$

$(x-3)(2x^2+3x-2)$

$\downarrow$ Factorise this

$(x-3)(x+2)(2x-1)$

Make $\dfrac{20x-35}{(x-3)(x+2)(2x-1)}$ into Partial Fractions.

$$\frac{20x-35}{(x-3)(x+2)(2x-1)} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{2x-1}$$

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$$20x-35 = A(x+2)(2x-1) + B(x-3)(2x-1) + C(x-3)(x+2)$$

let $x=3$

$$20(3)-35 = A(3+2)(2(3)-1) + 0$$

$$25 = A(5)(5)$$

$$25 = 25A$$

$$\boxed{A=1}$$

let $x=-2$

$$20(-2)-35 = A(0) + B(-2-3)(2(-2)-1) + 0$$

$$-75 = B(-5)(-5)$$

$$-75 = 25B$$

$$\boxed{B=-3}$$

let $x = 1/2$

$$20(1/2) - 35 = 0 + C(1/2 - 3)(1/2 + 2)$$

$$-25 = C(-5/2)(5/2)$$

$$-25 = -\frac{25}{4}C$$

$$C = 4.$$

$$\therefore \Rightarrow \frac{1}{x-3} - \frac{3}{x-2} + \frac{4}{2x-1}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x-3} - \frac{3}{x-2} + \frac{4}{2x-1}\,dx$$

$$\Rightarrow \ln|y| = \ln|x-3| - 3\ln|x-2| + \frac{4\ln|2x-1|}{2} \underbrace{+\ln A}_{\text{constant}}$$

$$\ln|y| = \ln\left|\frac{(x-3)\,(2x-1)^2}{(x-2)^3} \times A\right|$$

raise everything to e

$$y = \frac{A(x-3)(2x-1)^2}{(x-3)^2}$$